Higher-genus corrections to black-string solution

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Abstract.

One-string-loop (torus topology) corrections to black-string backgrounds corresponding to gauged $SL(2,R) \times R/R$ WZW model are calculated using β-function equations derived from string-loop-corrected effective action. Loop-corrected backgrounds are used to calculate ADM mass of the black string.

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1. Introduction

Recently much attention have recieved solutions of the gauged Wess-Zumino-Witten (WZW) G/H-models which provide conformal field theories interpreted as describing geometries of black holes, black strings, etc. [1, 2, 3, 4, 5, 6, 7, 8]. These solutions satisfy $O(\alpha')$ β-function equations [1], but there are also conformally exact results valid in all orders in α' [8, 10, 11]. However, usually these solutions are discussed for conformal field theories defined on manifolds of topology of the sphere, i.e. at the tree level of string-loop expansion. Some time ago, in papers [12], string-loop corrections to the tree-level solutions were discussed for 2D black-hole solution of gauged $SL(2,R)/U(1)$ WZW model. It was noted that in bosonic string theory, as a result of regularization of divergent integrals over the moduli, there appear imaginary corrections to the lowest genus solutions. Possible imaginary corrections to the mass of black hole could be interpreted as a manifestation of quantum instability of solution (stable at the tree level). However, in 2D theories the question of modular divergences is somewhat ambiguous because in this case modular divergences are absent. 2D theory can be considered as a limit from $D>2$-dimensional models which can have modular divergences, but in this setting the problem requires more careful analysis.

At present, a large variety of gauged WZW models was investigated which yield solutions interpreted as backgrounds of string theories in dimensions $D \geq 3$. In this paper, starting from loop-corrected renormalized string effective action (EA), we calculate loop corrections to tree-level backgrounds for the gauged $SL(2,R)\times R/R$ WZW model [2] which is the first one from the set of $SL(2,R)\times R^N/R$ models [3, 8] associated with $D=(N+2)$-dimensional backgrounds. Asymptotics of backgrounds are used to calculate string-loop-corrected ADM mass of the black string.

After fixing in sect.2 some notations, in sect.3 we introduce basic formulas of Tseytlin's approach to construction of string-loop-corrected EA. In sect.4 general expressions are applied to the one-loop case, i.e. for the torus topology. In sect.5 we calculate asymptotics of background solutions to β-function equations. In sect.6 these asymptotics are used to calculate loop-corrected ADM black-string mass. In sect.7 we discuss duality for the loop-corrected solutions of β- equations. Sect.8 contains concluding remarks and discussion.

2.

Gauged WZW models provide a natural framework for Lagrangian realization of coset models and form a bridge between conformal field theories and σ-model description of strings propagating in nontrivial backgrounds [1, 13]. For the gauged $SL(2,R)\times R/R$ WZW model, after setting the axial gauge and integrating out nonpropagating fields, in the limit of large central extension parameter k, in the leading order in $1/k$, one obtains the action [2]

$$I=\frac{k}{4\pi}\int d^2z\left\{-\left(1-\frac{1+\lambda}{r}\right)\partial t\bar\partial t+\left(1-\frac{\lambda}{r}\right)\partial x\bar\partial x+\frac{1}{4}\frac{\partial r\bar\partial r}{(r-\lambda)(r-1-\lambda)}+\right.$$
$$\left.+\sqrt{\frac{\lambda}{1+\lambda}}(1-\frac{1+\lambda}{r})\left(\partial x\bar\partial t-\bar\partial x\partial t\right)+\frac{1}{k}\sqrt{h}R^{(2)}(h)\Phi(r)\right\}. \qquad (1)$$

With identification $\frac{1}{k}\to\alpha'$, where α' is the string constant in dimensionless units, the

[1] See, however, paper [9] where backgrounds of $SL(2,R)\times R/R$ model were shown to satisfy β-function equations in $O(\alpha'^2)$ approximation.

action (1) can be interpreted as the action for the closed string propagating in 3D space-time equipped with the metric

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{1+\lambda}{r}\right)dt^2 + \left(1 - \frac{\lambda}{r}\right)dx^2 + \frac{1}{4}\frac{dr^2}{(r-\lambda)(r-1-\lambda)} \tag{2}$$

antisymmetric tensor gauge field

$$B_{tx} = \sqrt{\frac{\lambda}{1+\lambda}}\left(1 - \frac{1+\lambda}{r}\right) \tag{3}$$

and dilaton

$$\Phi = \frac{1}{2}(a - \ln r). \tag{4}$$

Backgrounds (2)-(4) are solutions of equations of motion derived from the $O(\alpha')$ part of string effective action (EA) [2, 3, 9]

$$S = a_0 \int d^D x \sqrt{|G|}\, e^{-2\Phi} \left[\Lambda - \frac{\alpha'}{2}\left(\hat{R} + 4D^2\Phi - 4(D\Phi)^2\right) + O(\alpha'^2)\right] \tag{5}$$

for $D = 3$. Here

$$\Lambda = \frac{D-26}{3}; \qquad \hat{R} = R - \frac{H^2}{12}$$

Equations of motion following from (5) are equivalent to conditions of Weyl invariance of the theory with the σ-model action

$$I = \frac{1}{4\pi\alpha'}\int d^2z\sqrt{h}\left[(G_{\mu\nu}h^{ab} + B_{\mu\nu}\frac{\varepsilon^{ab}}{\sqrt{h}})\partial_a x^\mu \partial^a x^\nu + \alpha' R^{(2)}\Phi(x)\right] \tag{6}$$

where h_{ab} is the world-sheet metric, and can be symbolically written as [14, 15]

$$\bar{\beta}^i(\varphi^i) = 0. \qquad (\varphi^i = G_{\mu\nu}, B_{\mu\nu}, \Phi) \tag{7}$$

3.

In closed bosonic string theory, the full string EA is obtained as the renormalized generating function of massless string amplitudes

$$S(\varphi_R) = \hat{Z}_R(\varphi_R) = \hat{Z}(\varphi(\epsilon), \epsilon)$$

where the bare fields $\varphi(\epsilon)$ are expressed as perturbative expansion in $\ln\epsilon$ with the coefficients depending on renormalized fields φ_R. Generating function $\hat{Z}(\varphi(\epsilon), \epsilon)$ is constructed as the sum of contributions from all the genera $\chi = 2 - n$

$$\hat{Z}(\varphi(\epsilon), \epsilon) = \frac{\partial}{\partial \ln\epsilon}\sum_{n=0}^{\infty}\bar{Z}_n(\varphi(\epsilon), \epsilon) \tag{8}$$

where n is the number of handles of the world-sheet surface [17, 18]. Here

$$\bar{Z}_n = \int [d\mu(\tau, \epsilon)]_n\, Z_n \tag{9}$$

and

$$Z_n = a_n \int D[x,h]' \sqrt{|\det G|} e^{-I}$$

is the partition function obtained by integration over all (functional) variables except for the moduli τ of the world-sheet metric h.

The basic assumption about $\hat{Z}(\varphi(\epsilon),\epsilon)$ is that it is perturbatively renormalizable both in α' and string-loop expansions [17, 18]. An important aspect of this property is that all divergences (modular and ultraviolet) are regularized in a universal way by the same cutoff parameter ϵ. The derivative with respect to $\ln\epsilon$ takes care of the properly accounted divergent volume of the Möbius group.

An explicit realization of such regularization is provided by Schottky parametrization [19, 20, 21, 18] of the extended moduli space. In this parametrization, a surface of genus $\chi = 2-n$ is mapped on the complex plane $\mathbf{C}$ [2] with n pairs of holes with the pairwise identified boundaries. On the complex plane $\mathbf{C}$ acts the group $SL(2,\mathbf{C})$. If the corresponding Möbius symmetry is not fixed, then the volume of the group $SL(2,\mathbf{C})$ enters the amplitudes as the universal divergent factor [3]. Fixing 3 complex parameters of the group $SL(2,\mathbf{C})$, one reduces the number of independent moduli to $3n-3$.

Divergences of the amplitudes can appear either if positions of several vertex operators (punctures) tend to each other or/and if the holes from the handles shrink to a point. In Schottky parametrization, all the divergences are universally regularized by introducing the "minimal distance" ϵ which enters propagators as well as integration measure over the moduli.

Partition function Z_n has the following form [17]

$$Z_n = a_n e^{\frac{\chi\Lambda}{2}\ln\epsilon} \int d^D x \sqrt{|G|}\, e^{-\chi\Phi} \left[1 + \alpha' \left(b_1^{(n)} \hat{R} + b_2^{(n)} D^2\Phi\right) + \dots\right], \tag{10}$$

where

$$b_1^{(n)} = \frac{\pi}{V}\int d^2z\sqrt{h}\, G(z,z') - \pi \int d^2z\sqrt{h}\, \left[\nabla_a \nabla'^a G(z,z')\, G(z,z') - (\nabla^a G(z,z'))^2\right]_{z=z'}$$

$$b_2^{(n)} = -\frac{1}{4}\int d^2z\sqrt{h}\, R^{(2)}(h)\, G(z,z) \tag{11}$$

$$V = \int d^2z\sqrt{h}.$$

Here $G(z,z')$ is the regularized propagator on the world sheet of genus $\chi = 2-n$ with a metric h (regularized Green function of scalar Laplacian).

The coefficients $b_{1,2}^{(n)}$ contain logarithmically divergent parts which appear from the limit of coinciding arguments in the propagators

$$b_1^{(n)} = \frac{1}{2}\ln\epsilon + \bar{b}_1^{(n)}; \quad b_2^{(n)} = (n-1)\ln\epsilon + \bar{b}_2^{(n)}, \tag{12}$$

and are defined up to transformations of the finite parts $\bar{b}_{1,2}^{(n)}$ under reparametrizations of the fields φ^i

$$G_{\mu\nu} \to G_{\mu\nu} + \alpha'(a_1 R_{\mu\nu} + a_2 G_{\mu\nu} R + a_3 D_\mu D_\nu \Phi + a_4 G_{\mu\nu} D^2\Phi + \dots) + \dots, \tag{13}$$

[2] More exactly, a surface is mapped on compactification of $\mathbf{C}$ i.e. on the 2-sphere [18].

[3] This is true for $n \geq 3$-point amplitudes.

$$\Phi \to \Phi + \alpha' b_1 R + \ldots, \quad B_{\mu\nu} \to B_{\mu\nu} + \alpha' c_1 D_\lambda H^\lambda_{\mu\nu} + \ldots$$

which do not change the massless sector of the (tree) string theory S-matrix [17, 18]. Renormalized partition function $Z_n^R(\varphi_R)$ is obtained by substituting expressions for bare fields in terms of renormalized fields. In the leading order in $\alpha' \ln \epsilon$ one has

$$\varphi^i = \varphi^i_R - \beta^i(\varphi_R) \ln \epsilon/\mu + \ldots . \tag{14}$$

4.

The tree-level (topology of sphere) generating functional $\bar{Z}_0$ contains no integration over the moduli. At the one-string-loop level (topology of torus), the "extended" moduli space is parametrized by three complex parameters ξ, η and k. The measure on the "extended" moduli space is

$$d\mu_1 = \frac{d^2\xi d^2\eta}{|\xi - \eta|^4}[d^2 k]. \tag{15}$$

Here $|\xi - \eta|$ has the meaning of the distance between the centers of the holes from the handle on the complex plane $\mathbf{C}$ [4]. In parametrization $k = e^{2\pi i \tau}$, the measure $[d^2 k]$ is given by

$$[d^2\tau] = \frac{d^2\tau}{\tau_2^2}\left(\tau_2 \left|\eta(\tau)\right|^4\right)^{-\frac{D-2}{2}}, \tag{16}$$

where $\eta(\tau)$ is the Dedekind η-function $\eta(\tau) = k^{\frac{1}{24}} \prod_1^\infty (1 - k^m)$. Note that $k \sim e^{-2\pi\tau_2}$ as $\tau_2 \to \infty$. Propagator on the complex plane $\mathbf{C}$ with two discs from the handle removed is

$$G(z_1, z_2) = -\frac{1}{4\pi} \ln\left\{\left(\left|z_1 - z_2\right|^2 + \epsilon^2\right) \prod_{m=1}^{\infty} \frac{(1-\lambda k^m)(1-\lambda^{-1}k^m)}{(1-k^m)^2}\right\} + \frac{(\ln|\lambda|)^2}{2\pi\tau_2}, \tag{17}$$

where

$$\lambda = (z_1 - \xi)(z_2 - \eta)(z_1 - \eta)^{-1}(z_2 - \xi)^{-1}.$$

Singularity at $\xi = \eta$ in the measure (15) is regularized by the same cutoff as in the propagator (17): $|\xi - \eta|^2 \to |\xi - \eta|^2 + \epsilon^2$.

Performing integrations in the formulas (11), one obtains the generating functional $\bar{Z}_1$ in the form [17, 18]

$$\bar{Z}_1 = a_1 \int d^D y \sqrt{|G|} \int [d^2\tau] \left[\ln \epsilon \left(1 + \frac{\alpha'}{2}\hat{R} \ln \epsilon + \alpha'(b_1^{(1)}\hat{R} + b_2^{(1)}(D\Phi)^2)\right) + O(\alpha'^2)\right]. \tag{18}$$

Here the first logarithmic factor appears from integration over the moduli, the second one is due to ultraviolet divergences in the propagators at the coinciding arguments. All constant factors are included in a_1. [5]

Collecting tree-level and one-loop contributins, one has

$$\begin{aligned} \hat{Z}(\varphi(\epsilon), \epsilon) \quad &= \quad a_0 e^{\Lambda \ln \epsilon} \int d^D x \sqrt{|G|}\, e^{-2\Phi} \left[\Lambda - \frac{\alpha'}{2}\left(\hat{R} + 4(D\Phi)^2\right) + O(\alpha'^2)\right] + \qquad (19) \\ & a_1 \int [d^2\tau] \int d^D x \sqrt{|G|}\, [1 + \alpha' \hat{R} \ln \epsilon + \alpha'(b_1^{(1)}\hat{R} + b_2^{(1)}(D\Phi)^2) + O(\alpha'^2)] \end{aligned}$$

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[4] To be precise, the measure (14) can be used to calculate $N \geq 3$-point amplitudes. To calculate $N \leq 3$-point amplitudes and, in particular, the vacuum functional $\bar{Z}_1$ some modifications are required [18] yielding the final expression (18).

[5] Substituting propagator (17) in expressions (11), we obtain that $\bar{b}_1^{(1)} = \bar{b}_2^{(1)} = 0$. However, nonzero terms $\bar{b}_{1,2}^{(1)}$ can be generated by transformations (13).

Renormalized generating functional $\hat{Z}_R$ is obtained by substituting expressions for bare fields in terms of renormalized ones and taking into account additional terms from string-loop divergences (cf. with (13)) [17, 18].

$$\varphi^i = \varphi^i_R - \beta^i(\varphi_R)\alpha' \ln \epsilon/\mu + \delta\beta^i(\varphi_R)\alpha' \ln \epsilon/\mu + \dots \tag{20}$$

Additional terms $\delta\beta^i(\varphi_R)$ are obtained from the requirement to cancel string-loop $\ln \epsilon$ terms in (18) and are equal to

$$\delta\beta^i = \rho e^{2\Phi^R} \left(\frac{d}{4}, \quad \frac{1}{2} g^R_{\mu\nu}, \quad B^R_{\mu\nu} \right), \tag{21}$$

where $\rho = 2a_1/a_0 \int [d^2\tau]$. Substituting (20) in (19), we obtain the expression for the renormalized generating functional $\hat{Z}_R$ which includes contributions from sphere and torus topologies

$$\begin{aligned} \hat{Z}_R &= a_0 \int d^D x \sqrt{|G|} e^{-2\Phi} \left[\Lambda - \frac{\alpha'}{2} \left(\hat{R} + 4(D\Phi)^2 \right) + \right. \\ & \left. \frac{\rho}{2} e^{2\Phi} \left(1 + \alpha' \hat{R} \ln \mu + \alpha' (b_1^{(1)} \hat{R} + b_2^{(1)} (D\Phi)^2) \right) + \dots \right] \end{aligned} \tag{22}$$

(henceforth we omit the subscript R). It is seen that the formal effective parameter of string-loop expansion is $\rho e^{2\Phi}$. Using the freedom in the choice of reparametrization of fields (13) the terms $\alpha' \hat{R} \ln \mu + \alpha' (b_1^{(1)} \hat{R} + b_2^{(1)} (D\Phi)^2)$ can be set to zero. Adding to the action (22) the total derivative $2D^2(e^{-2\Phi})$ to have the same tree-level part of the action as in (5), one finally obtains the renormalized EA

$$S = a_0 \int d^D x \sqrt{|G|} e^{-2\Phi} \left[\Lambda - \frac{\alpha'}{2} \left(\hat{R} + 4D^2\Phi - 4(D\Phi)^2 \right) + \frac{\rho}{2} e^{2\Phi} \right]. \tag{23}$$

5.

Our next aim is to find string-loop corrections to tree-level solutions of eqs. (6). Variation of EA (22) yields the following equations of motion:

$$\begin{aligned} \bar{\beta}^G_{\mu\nu} + \delta\bar{\beta}^G_{\mu\nu} &= \hat{R}_{\mu\nu} + 2D_\mu D_\nu \Phi + \frac{\rho}{2\alpha'} G_{\mu\nu} e^{2\Phi} = 0, & (24) \\ \bar{\beta}^\Phi &= \hat{R} - \frac{2\Lambda}{\alpha'} - 4(D\Phi)^2 + 4(D^2\Phi) = 0, & (25) \\ \bar{\beta}^B_{\mu\nu} &= D^\lambda \Phi H_{\lambda\mu\nu} = 0. & (26) \end{aligned}$$

Tree-level backgrounds (2)-(4) depend on a single parameter r and provide an example of the "rolling moduli" solution [26] to non-linear eqs. (7). In the following, having in view calculation of the mass of the black string, we shall be interested in finding asymptotics of solutions to nonlinear equations (24)-(26) at $r \to \infty$. In this limit, we can linearize the system (24)-(26) and solve it explicitly.

Solving the tree-level eqs. (24)-(26) (with $\rho = 0$), one obtains the "rolling" solution for the metric and dilaton (together with the corresponding solution for the antisymmetric tensor) of the form

$$ds^2 = -\frac{\sinh^2 \frac{\gamma z}{2}}{\lambda + \cosh^2 \frac{\gamma z}{2}} dt^2 + \frac{\cosh^2 \frac{\gamma z}{2}}{\lambda + \cosh^2 \frac{\gamma z}{2}} dx^2 + dz^2, \tag{27}$$

$$\Phi = \frac{1}{2}(a - \ln\cosh^2\frac{\gamma z}{2})$$

where $\gamma^2 = \frac{2|\Lambda|}{\alpha'}$. Taking $\gamma^2 = 4$ (α' in dimensionless units), and introducing new variable r by the relation

$$r = \lambda + \cosh^2 z$$

one obtains the solution (2)-(4) of the gauged WZW model. The metric and dilaton (27) are asymptotic to the flat-space solution $(\eta_{\mu\nu},\, \Phi^0)$, where $\Phi^0 = \frac{1}{2}(a - \gamma|z|)$. Flat-space solution can be considered as the limiting form of the "rolling" solution (27) as $\alpha' \to 0 \quad (\gamma \to \infty)$ [6].

To solve loop-corrected equations, let us introduce new variable z by the relation $r = \lambda + \cosh^2\frac{\gamma z}{2} + f(\rho, z)$, where the function f is chosen so that in new variables the zz component of the metric is again equal to unity. Asymptotically as $z \to \infty, \quad f(z) = O(\rho z^n e^{-\gamma|z|})$ with some n. As in the case $\rho = 0$, we are looking for solution for the metric and dilaton asymptotic to the flat-space solution. Writing the metric and dilaton as

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\Phi = \Phi^0 + \varphi,$$

where $h_{\mu\nu}$ and φ are of order $O(\rho z^n e^{-\gamma|z|})$ and linearizing the equations about the flat-space solution, we obtain

$$h''_{tt} - 2\Phi^{0\prime} h'_{tt} + \frac{\rho}{\alpha'} e^{2\Phi^0} = 0 \tag{28}$$

$$h''_{xx} - 2\Phi^{0\prime} h'_{xx} - \frac{\rho}{\alpha'} e^{2\Phi^0} = 0$$

$$h''_{tt} - h''_{xx} + 4\varphi'' + \frac{\rho}{\alpha'} e^{2\Phi^0} = 0 \tag{29}$$

$$h''_{\mu\nu} - 2\Phi^{0\prime} h'_{\mu\nu} = 0 \qquad (\mu \neq \nu). \tag{30}$$

Here primes stand for derivatives with respect to z. The term $H^2_{\mu\nu}$ is asymptotically of order $O(e^{-2\gamma|z|})$ and can be neglected as a small correction to the leading terms. Integrating eqs. (28) we get

$$h'_{tt} = e^{2\Phi^0}\left(-\frac{\rho}{\alpha'} z + c_t\right) \tag{31}$$

$$h'_{xx} = e^{2\Phi^0}\left(\frac{\rho}{\alpha'} z + c_x\right) \tag{32}$$

and

$$h'_{\mu\nu} = e^{2\Phi^0} c_{\mu\nu} \tag{33}$$

The constants c_t and c_x are assumed to be independent of ρ and can be defined by taking the limit $\rho = 0$ and comparing with the tree-level solution. In the same way, assuming that the constants $c_{\mu\nu}$ are ρ-independent and comparing (33) with the tree-level solution, we set $c_{\mu\nu} = 0$. Integrating eq. (29) we get

$$h'_{tt} - h'_{xx} + 4\varphi' + \frac{\rho}{\alpha'}\int dz e^{2\Phi^0} = const.$$

[6] Dilaton Φ^0 is composed from two branches of solutions to the flat-space dilaton equation $\frac{\gamma^2}{4} = (\Phi^{0\prime})^2$. In the following, this results in "constants" having the sgn$z$ factor. Note that we are interested only in asymptotic region of large $|z|$ where, up to exponentially small terms, one can use a smooth approximation of Φ^0, for example, Φ from (27).

Adjusting the const to have asymptotically vanishing solution, we obtain

$$h'_{tt} - h'_{xx} + 4\varphi' = -\frac{\rho}{\alpha'\gamma}\mathrm{sgn} z e^{a-\gamma|z|} \tag{34}$$

On the other hand, linearizing eq. (25) about the flat-space solution, we have

$$h''_{tt} - h''_{xx} - \frac{2\Lambda}{\alpha'} - 4\left(\Phi^{0\prime}\right)^2 - 8\Phi^{0\prime}\varphi' + 4\varphi'' - 8\Phi^{0\prime}(h'_{tt} - h'_{xx}) = 0. \tag{35}$$

Noting that

$$-\frac{2\Lambda}{\alpha'} - 4(\Phi^{0\prime})^2 = 0$$

is the equation for the vacuum dilaton, equation (35) can be rewritten in a form

$$\left(\frac{d}{dz} - 2\Phi^{0\prime}\right)(h'_{tt} - h'_{xx} + 4\varphi') = 0$$

and solved as

$$h'_{tt} - h'_{xx} + 4\varphi' = ce^{2\Phi^0} \tag{36}$$

Comparing (34) and (36), we see that both forms of solution are equivalent up to exponentially small corrections if we make the identification $c = -\frac{\rho}{\alpha'\gamma}\mathrm{sgn} z$.

6.

Having obtained loop-corrected asymptotics of backgrounds, we can calculate string-loop-corrected mass of black string. In standard gravity interacting with matter, for a class of metrics which asymptotically sufficiently quickly approach the flat-space metric, the total energy of a field configuration can be defined in the framework of canonical approach [22, 23, 24] [7]. The total energy is defined as the value of the hamiltonian taken on the shell of zero constraints $\{\Psi\} = 0$. The resulting expression is of the form of space integral over the total derivative

$$E = -\frac{1}{\kappa_D}\int d^{D-1}x \partial_i(\sqrt{|G_{D-1}|}f^i)|_{\{\Psi\}=0}, \tag{37}$$

where

$$f^i = G_{lm,k}(E^{il}E^{km} - E^{ik}E^{lm})$$

Here G_{ik} is the spacial $(D-1)$-dimensional part of the metric, $G_{D-1} = \det G_{ik}$, and $E^{ik}G_{kl} = \delta^i_l$. For solutions with $G_{0i} = 0$ this formula is simplified because in this case $E^{ik} = G^{ik}$.

In dilatonic gravity, separating the α' dependence in $a_0 = N(\alpha')^{-D/2}$ and introducing the D-dimensional gravitation constant

$$\frac{1}{\kappa_D} = \frac{N}{2}(\alpha')^{-\frac{D-2}{2}},$$

the action (22) is written as

$$S = -\frac{1}{\kappa_D}\int d^D y \sqrt{|G|} e^{-2\Phi}\left[\hat{R} - 4(D\Phi)^2 + 4D^2\Phi - \frac{2\Lambda}{\alpha'} + \frac{\rho}{\alpha'}e^{2\Phi}\right]. \tag{38}$$

[7] It should be mentioned that in the absence of a preferred asymptotic frame, the notions of ADM energy and mass are nonunique. As usual, we introduce energy as a quantity conjugate to variable t.

For the genus zero part of the action ($\rho = 0$), for solutions which sufficiently rapidly approach the vacuum solution ($\eta_{\mu\nu}, \Phi^0$), calculations similar to those in the standard gravity give [25]

$$E = -\frac{1}{\kappa_D} \int d^{D-1}x \partial_i \left[e^{-2\Phi} \sqrt{|G_{D-1}|} (f^i - 4G^{ik}\partial_k \varphi) \right] |_{\{\Psi\}=0}, \tag{39}$$

Here we again assumed that $G_{0k} = 0$. The expression (39) is valid also for the action (38) containing the one-string-loop correction, because the latter contributes only to the potential part of the action (38).

In the case $D = 3$, for solutions sufficiently rapidly approaching the vacuum configuration, the divergence in the integrand in (39) is asymptotically equal to

$$\left[e^{-2\Phi^0} (h'_{xx} - 4\varphi') \right]'.$$

Coordinate x asymptotically measures distances along the string and z is the transverse coordinate. Substituting the asymtotic expression for the black string solution (34) in (39) we obtain the mass of the black string per unit length

$$E = -N(\alpha')^{1/2} \left(e^{-2\Phi^0} h'_{tt} - \frac{\rho}{\alpha' \gamma} \right) |_{z \to \infty}. \tag{40}$$

For the tree-level solution ($\rho = 0$), the expression (40) reproduces the the mass of the black string calculated in ref. [2]:

$$E|_{\rho=0} = \frac{8N}{\sqrt{k}} (1 + \lambda) e^{-a}$$

where we substituted $\alpha' = \frac{1}{k}$.

If string-loop correction is taken into account, h'_{tt} contains a term linear in z, and expression (40) diverges, the divergence being proportional to ρ. This means that, modifying the tree-level bosonic string action by one-loop corrections, one cannot define finite ADM mass for the black-string solution. If one defines the energy by subtracting the infinite part which is proportional to the mixing parameter ρ, one again obtains an expression independent of ρ.

7.

It is well known that the $3D$ black-string solution is dual to the spherically-symmetric $3D$ black hole solution [27, 28, 29, 30]. To have the dual solution in the standard form, solution (2)-(4) is written as

$$ds^2 = -\left(1 - \frac{r_+^2}{r^2}\right) d\hat{t}^2 + \left(1 - \frac{r_-^2}{r^2}\right) d\hat{x}^2 + \left(1 - \frac{r_+^2}{r^2}\right)^{-1} \left(1 - \frac{r_-^2}{r^2}\right)^{-1} \frac{l^2 dr^2}{r^2} \tag{41}$$

$$B_{\hat{x}\hat{t}} = \frac{r_+ r_-}{r^2}, \qquad \Phi = \frac{1}{2}(a - \ln r^2).$$

In new variables

$$t = al(\hat{x} - \hat{t}), \quad \varphi = a(r_+^2 \hat{t} - r_-^2 \hat{x}), \quad a = (r_+^2 - r_-^2)^{-1/2}$$

it takes the form

$$ds^2 = -\left(M - \frac{J^2}{4r^2}\right)dt^2 + \frac{2}{l}dtd\varphi + \frac{1}{r^2}d\varphi^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1}dr^2 \qquad (42)$$

$$B_{\varphi t} = -\frac{J}{2r^2}, \qquad \Phi = \frac{1}{2}(a - \ln r^2).$$

where

$$M = \frac{r_+^2 + r_-^2}{l^2}, \qquad J = \frac{2r_+r_-}{l}$$

Dual (with respect to the variable φ) transformation of the fields (42) gives the solution

$$d\tilde{s}^2 = -\left(M - \frac{r^2}{l^2}\right)dt^2 + Jdtd\varphi + r^2d\varphi^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1}dr^2 \qquad (43)$$

$$B_{\varphi t} = -\frac{r^2}{l^2}, \qquad \Phi = 0.$$

The metric $d\tilde{s}^2$ is the black-hole solution in $3D$ Einstein gravity [27, 28, 29, 30]. The fields (43) are solutions of equations of motion for the action (5) with the cosmological constant $\Lambda = -\frac{2\alpha'}{l^2}$. Duality transformation [31] leaves the form of equations of motion unchanged.

Let us consider duality transformations in the theory with the loop-corrected action (23). Requiring that equations of motion (24)-(26) do not change their functional form under duality transformations (cf. [31]) $\varphi^i = \varphi(\tilde{\varphi}) + \rho\delta\varphi(\tilde{\varphi})$, we have

$$\beta^i(\varphi) + \rho\delta\beta^i(\varphi)|_{\varphi^i=\varphi(\tilde{\varphi})+\rho\delta\varphi(\tilde{\varphi})} = \beta^i(\tilde{\varphi}) + \rho\delta\beta^i(\tilde{\varphi}).$$

Keeping terms linear in ρ, we obtain

$$\delta\beta + \delta\varphi^j\frac{\partial\beta^i}{\partial\varphi^j}|_{\varphi^i=\varphi^i(\tilde{\varphi})} = \delta\beta^i(\tilde{\varphi})$$

The number of equations on the functions $\delta\varphi^i$ is equal the number of the functions $\delta\varphi^i$. Finding the functions $\delta\varphi^i$, we obtain the duality transformations which leave the functional form of the loop-corrected β-equations unchanged.

8. Conclusions and discussion.

In this paper, starting from string-loop-corrected renormalized EA, we calculated one-string-loop corrections to black-string backgrounds which, on one hand, are obtained from the gauged WZW model, and, on the other hand, are solutions to $O(\alpha')$ β-function equations derived from tree-level EA. Although final calculations were performed for an example of $SL(2,R) \times R/R$ WZW model, our discussion was quite general: all the expressions can be written in D-dimensional form and applied to the case of a general $SL(2,R) \times R^N/R$ model. It was found that backgrounds acquire corrections of order $\rho\int[d^2\tau]$, where ρ is parameter accounting for an admixture of genus-one contribution to the tree-level part of EA.

From (15) it follows that for all $SL(2,R) \times R^N/R$ models and, in particular, for the $3D$ black-string solution, the integral over the moduli is exponentially divergent. This divergence is the well-known tachyonic divergence in bosonic string theory, which is absent in superstring theory. In paper [32] this problem was discussed in the framework of

fermionic string theory and it was argued that in this theory there appear no terms in EA which could give divergent corrections to tree-level result.

Using the one-string-loop-corrected backgrounds obtained by solving the β-equations, we calculated the ADM mass of the black string. It appeared that the result is divergent, the divergence being proportional to the mixing parameter ρ. Redefining the energy (mass) by subtracting the infinite part, one obtains an expression independent of ρ. Thus, in the present case, the conjecture of ref. [12] about the imaginary string-loop corrections to the tree-level mass does not work.

Although our calculations were restricted to one-loop contributions, higher- order corrections can be discussed as well. Loop-corrected EA has the following structure:

$$
\begin{gathered}
\hat{Z} = \hat{Z}_0 + \hat{Z}_1 + \hat{Z}_2 + \ldots = \\
a_0 \int d^D x \sqrt{G} e^{-\Phi}(-2\Lambda + \alpha' R + \ldots) + a_1 \int d^D x \sqrt{G}(1 + \ldots) + \\
a_2 \int d^D x \sqrt{G} e^{\Phi}(1 + \ldots) + \ldots
\end{gathered}
\tag{44}
$$

(here a_i include integrals over the moduli). It is seen that higher-genus corrections are accompanied by the factors $e^{\frac{\chi\Phi}{2}}$ and are exponentially suppressed at spatial infinity for the black-string solution in question. Thus, in any finite order in string-loop expansion, corrections from higher topologies will not contribute to the ADM mass.

For the $SL(2,R)/R$ model, loop corrections can be calculated in the same way as above. However, in this case, because of 2D relation $R_{\mu\nu} = \frac{1}{2} G_{\mu\nu} R,$ β-equations are much simpler and can be easily solved exactly. In notations of [33] we have

$$\Phi = \frac{Q\eta}{2}$$

$$G_{\mu\nu} = diag[-g(\eta), g(\eta)^{-1}]$$

where $g(\eta)$ is now solution of loop-corrected equation

$$g'' = Qg' + \frac{\rho}{\alpha'} e^{Q\eta} \tag{45}$$

of the form

$$g(\eta) = 1 + a e^{Q\eta} + \frac{\rho\eta}{\alpha' Q} e^{Q\eta}.$$

Note that in this case, as in $3D$ theory there appear the term $O(\eta e^{Q\eta})$. The relation $Q^2 = |\Lambda|$ does not recieve loop corrections and is the same as at the tree level. Since in 2D theory there is no tachyon in the spectrum, integration measure over the moduli contains no exponential factors.

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